Math 102

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October 16, 2018

Announcements

- Review Sessions for Thursday 10/25 Midterm
  - Monday 10/22 in Buchanan A201, 3-7pm
  - Tuesday 10/23 in CHBE 101, 3-7pm
- **Reminder:** OSH deadlines are at 9pm.
  - If you have an issue with submission, **send me a message with your OSH by before 12 midnight.**

Goals Today

- Optimal Foraging - optimizing long-term average rate of gain subject to constraints.
- Least Squares - using optimization to fit a set of data points with a best-fit trend line

OSH 3 Recap

Tradeoff between Math and Chemistry depends on difficulty of math exam, k.
https://www.desmos.com/calculator/73d5om1uwo

Recall: from OSH 2

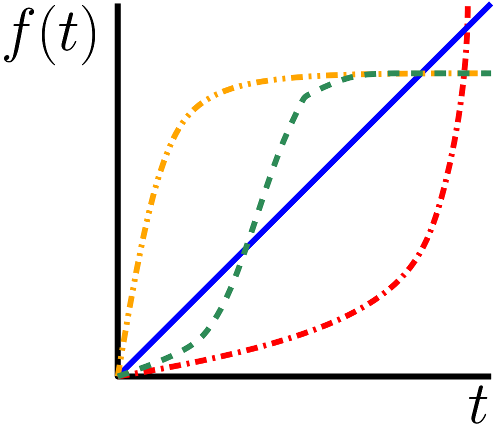
Bears search for berries in patches that grow in patches that can be spread out across a large area. A bear will spend time in one patch gathering food before moving to another patch. The number of berries collected in a patch depends on the amount of time spent in the patch:

$$f(t) = \frac{At}{k+t}$$

https://www.desmos.com/calculator/cwkxvbdpj2

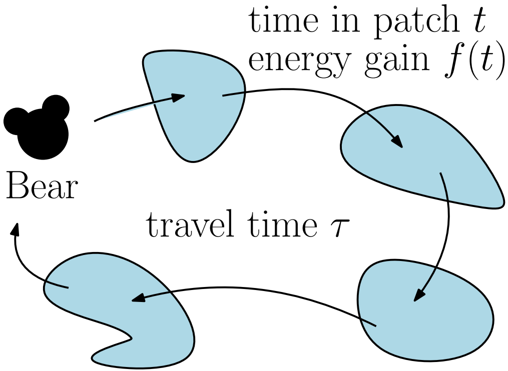
$f(t)$ = Total food collected

Question: For each graph, give a verbal description of how the **food collection rate** behaves over time.



How long should I spend per food patch?

- Stay too long in one patch - diminishing returns as the patch gets depleted.
- Leave a patch too early - spend unnecessary time traveling.



Optimal Foraging

Maximize average rate of **food collected per unit time**

$$R(t) = \frac{\text{food collected per patch}}{\text{travel time} + \text{time in patch}} = \frac{f(t)}{\tau + t}$$

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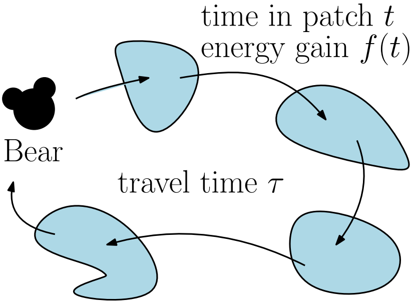
- Bear can't control $\tau =$ travel time per patch. Treat this as a constant. Same with A and k.
- Bear can control $t =$ time spent per patch. Treat this as a variable.



$$R(t) = \frac{At}{(\tau + t)(k + t)}$$

'If the bear chooses to spend t minutes in each patch, the overall **average rate** of food collection will be $\frac{At}{(\tau+t)(k+t)}$.'

Goal: Maximize $R(t)$, i.e. maximize efficiency.

Sketch $R(t) = \frac{At}{(\tau+t)(k+t)}$

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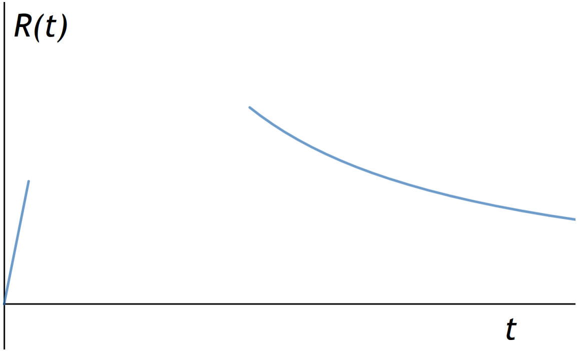

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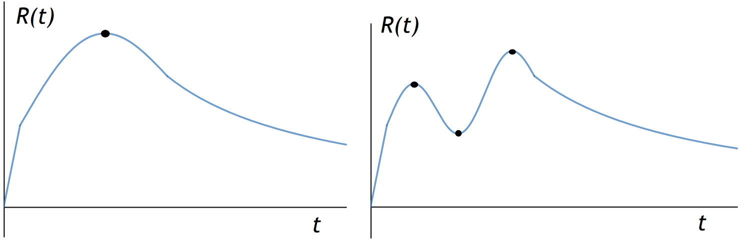

To figure out which sketch is accurate, let's find the **critical points.**

Maximize $R(t) = \frac{At}{(\tau+t)(k+t)}$

Find the critical points

$$R'(t) = A\frac{k\tau - t^2}{(k+t)^2(\tau+t)^2} = 0$$

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$$\Longrightarrow \; k\tau - t^2 = 0 \Rightarrow t = \pm\sqrt{k\tau}$$

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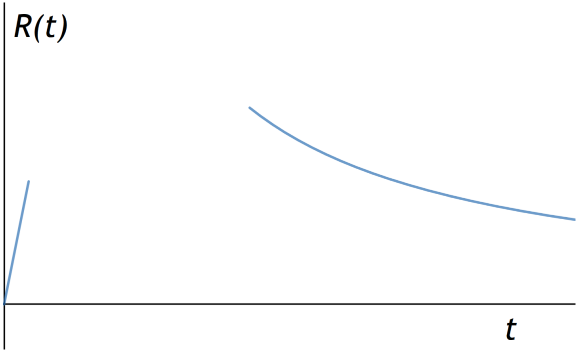
$$\implies k\tau - t^2 = 0 \Rightarrow t = \pm\sqrt{k\tau}$$

Keep the positive root: $\boxed{t = \sqrt{k\tau}}$.

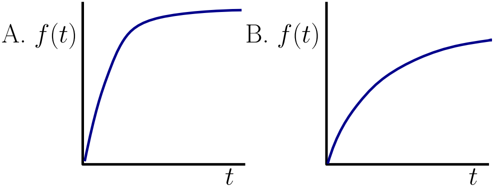
CP at $t = \sqrt{k\tau}$

Question: Give an airtight argument that $t = \sqrt{k\tau}$ is the global maximum of $R(t)$. (Hint: Think about **sketching the derivative** $R'(t)$)



Bear eating berries

Question: The graph on the left shows $f(t)$ in summer, while the graph on the right shows $f(t)$ in autumn.



$f(t) = \frac{t}{20+t}$ $\quad$ $f(t) = \frac{t}{200+t}$

Should the bear spend more time per patch in summer, or in autumn?

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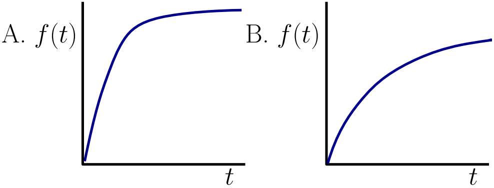


$f(t) = \frac{t}{20+t}$ $\quad$ $f(t) = \frac{t}{200+t}$

Should the bear spend more time per patch in summer, or in autumn? Autumn - larger $k \implies \sqrt{k\tau}$ is larger.

Another application - Optimal studying

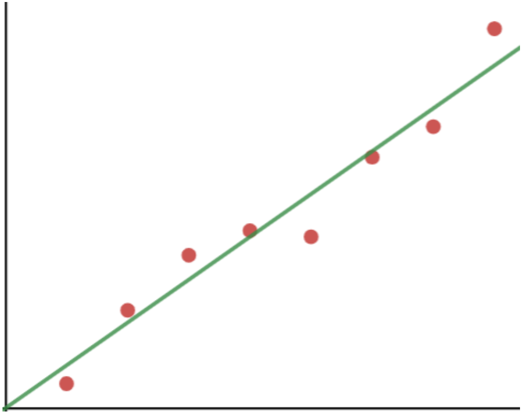
- As you study a subject, you get tired. Your efficiency decreases. Knowledge gained after t minutes of study: $f(t)$.
- Length of study break: τ.

Long-term average rate of knowledge gained if you spend t minutes before taking breaks:

$$R(t) = \frac{f(t)}{\tau + t}$$

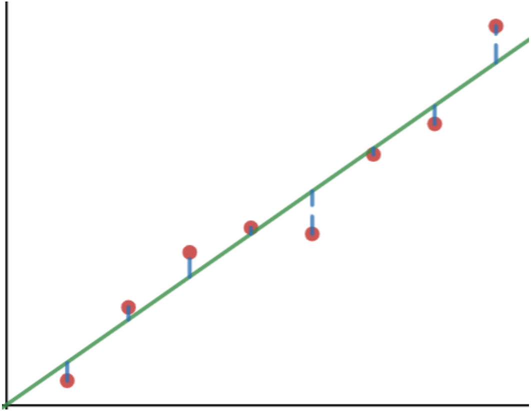
Data-fitting

Question: Given a collection of observed data points, how to find the best-fit line?

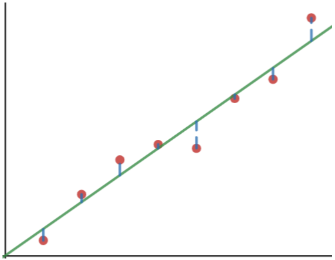


Data-fitting

Approach: Consider all possible lines, and pick the one which minimizes error.



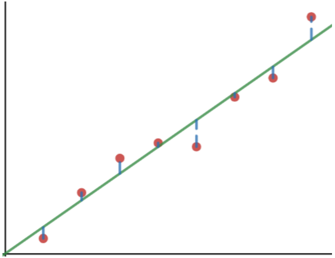
Residuals



Points:
$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Line: $y = ax$ for some a

Residuals



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$$r_i = ax_i - y_i$$

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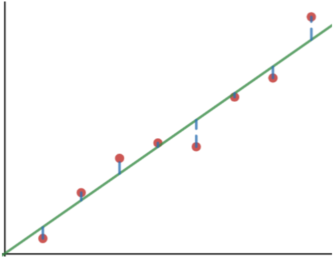

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Approach: Minimize the sum of squared residuals

$$r_1^2 + \ldots + r_n^2 = (ax_1 - y_1)^2 + \ldots + (ax_n - y_n)^2$$

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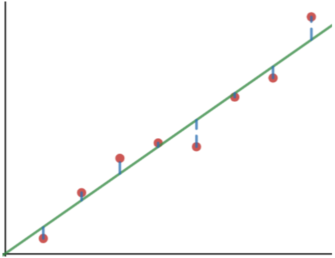

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Question: What is the **variable** here?

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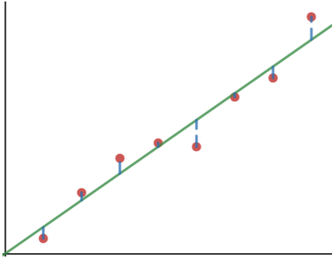

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Residuals: i-th residual is

$$r_i = ax_i - y_i$$

Approach: Minimize the sum of squared residuals

$$r_1^2 + \ldots + r_n^2 = (ax_1 - y_1)^2 + \ldots + (ax_n - y_n)^2$$

Question: What is the **variable** here? Answer: a

$$SSR(a) = (ax_1 - y_1)^2 + \ldots + (ax_n - y_n)^2$$

Find the critical points:

$$SSR'(a) = 2(ax_1 - y_1)x_1 + \ldots + 2(ax_n - y_n)x_n = 0$$

$$(ax_1 - y_1)x_1 + \ldots + (ax_n - y_n)x_n = 0$$

$$ax_1^2 + ax_2^2 + \ldots + ax_n^2 = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

$$\boxed{a = \frac{x_1y_1 + x_2y_2 + \ldots + x_ny_n}{x_1^2 + x_2^2 + \ldots + x_n^2}}$$

Exercise: Calculate the second derivative, $SSR''(a)$, to show that this is a minimum!

Example

i	1	2	3	4
x_i	2	3	5	6
y_i	1.5	2	3	4.5

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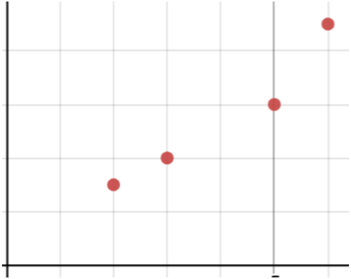

$$a = \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

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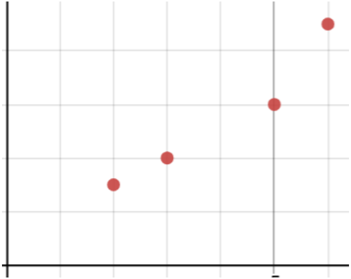

$$\begin{aligned} a &= \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \\ &= \frac{3 + 6 + 15 + 27}{4 + 9 + 25 + 36} \\ &= \frac{51}{74} \end{aligned}$$

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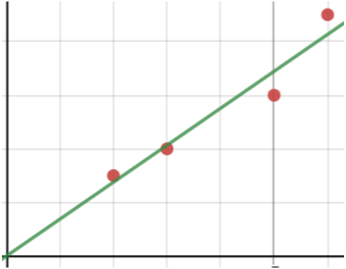

$$a = \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \frac{3 + 6 + 15 + 27}{4 + 9 + 25 + 36} = \frac{51}{74}$$

The best-fit line is $y = \frac{51}{74}x$.

Recap

- Optimal Foraging - optimizing long-term average rate of gain subject to constraints.
- Least Squares - using optimization to fit a set of data points with a best-fit trend line